JUNE 20, 2007 MAT1190 FINAL EXAM Total Points=224+5 Bonus

NAME ____________ ~~STUDENT ID~~ ____ Key.

1. Simplify.
[15 points]

a) $2^{-1}+2^0 = \frac{1}{2}+1 = \frac{1}{2}+\frac{2}{2} = \frac{3}{2}$

b) $\dfrac{\frac{1}{6a^2b^3}}{\frac{5}{8a^3b}-\frac{3}{10a^3b^4}} = \frac{1}{6a^2b^3} \div \left[\frac{5}{8a^3b}-\frac{3}{10a^3b^4}\right] =$

$\frac{1}{6a^2b^3} \div \left[\frac{25b^3-12}{40a^3b^4}\right] = \left(\frac{1}{6a^2b^3}\right)\cdot\left(\frac{40a^3b^4}{25b^3-12}\right) = \frac{20ab}{3(25b^3-12)}$

c) $3x^2-x-2-(-x^2+2x-5) = 3x^2-x-2+x^2-2x+5$

$= 4x^2-3x+3$

d) $\frac{x-1}{x-2}-\frac{x+3}{x+4} = \frac{(x-1)(x+4)}{(x-2)(x+4)} - \frac{(x+3)(x-2)}{(x+4)(x-2)}$

$= \frac{x^2+3x-4-(x^2+x-6)}{(x-2)(x+4)} = \frac{2x+2}{(x+4)(x-2)}$

e) $-(-|-2-5|)$

$-(-|-7|) = -(-(7)) = 7$

2. Factor completely
[12 points]

a) $6x^3-6x =$

$6x(x^2-1) = 6x(x-1)(x+1)$

b) y^4-625

$(y^2+25)(y^2-25) = (y^2+25)(y-5)(y+5)$

c) $9h^2+24ht+16t^2 = (3h+4t)(3h+4t)$

d) $x^3y^4+4x^2y^2-12x = x(x^2y^4+4xy^2-12)$

$x(xy^2+6)(xy^2-2)$

3. Let $f(x)=e^x$, $g(x)=\log_2(x)$, and $h(x)=\ln(x)$. Evaluate and simplify the following.
[18 points]

a) $g(64)\cdot h(e^3) = \left[\log_2 64\right]\left[\ln(e^3)\right] = (6)(3) = 18.$

b) $f(h(4)) = f(\ln 4) = e^{\ln 4} = 4.$

c) $g^{-1}(4)$; $g^{-1}(x) = 2^x$, $g^{-1}(4) = 2^4 = 16$

d) $h(e)\cdot g\left(\frac{1}{2}\right) = (\ln(e))\cdot\left(\log_2 \frac{1}{2}\right) = 1\cdot(-1) = -1$.

e) $f^{-1}(e)$; $f^{-1}(x) = \ln x$, $f^{-1}(e) = \ln e = 1$.

f) $f(\ln 5) = e^{\ln 5} = 5.$

4. Is the following function invertible? Why or why not?
[5 points]

t	-1	0	1	5	7	8
$f(t)$	7	5	3	4	0	3

$f(1)=3$

$f(8)=3$. The function is not one-to-one.

. So $f(t)$ is not invertible.

5. Find the average rate of change of $f(x)=5x^3-7$ between $x=-1$ and $x=2$.
[4 points]

$$= \frac{f(2)-f(-1)}{2-(-1)} = \frac{(5(2)^3-7)-(5(-1)^3-7)}{3}$$

$$= \frac{45}{3} = 15$$

6. Are the lines $y=7x+3$ and $y=-7x+3$ perpendicular, parallel or neither?
[3 points]

$m_1 = \text{slope} = 7$, $\text{slope} = -7 = m_2$

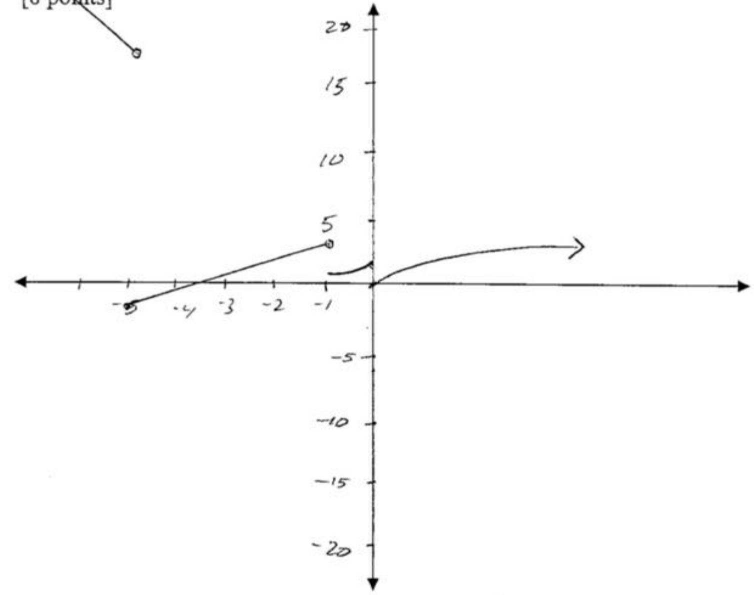
1) $m_1 \neq m_2 \Rightarrow$ not parallel.

2) $7(-7) \neq -1 \Rightarrow$ not perpendicular.

7. Given the following function:

$$f(x)=\begin{cases}-2x+7, & x<-5\\ x+4, & -5\le x<-1\\ 2^x, & -1\le x\le 0\\ \sqrt[3]{x}, & x>0\end{cases}$$

a) Sketch the graph of $f(x)$. Clearly label all endpoints.
[8 points]



b) Evaluate $f(f(-5))$. $= f(-1) = 2^{-1} = \frac{1}{2}$
[3 points]

c) On what intervals is $f(x)$ increasing?
[3 points] $[-5,-1),\ [-1,0],\ (0,\infty)$

8. Find the inverse of the following functions.
[10 points]

a) $f(x)=\dfrac{\sqrt{x}}{\sqrt{x}+1}$, $y=\dfrac{\sqrt{x}}{\sqrt{x}+1}$

$x(\sqrt{y}+1)=\sqrt{y}$

$x\cdot\sqrt{y}+x=\sqrt{y}$

1) $x=\dfrac{\sqrt{y}}{\sqrt{y}+1}$

$\sqrt{y}(x-1)=-x$

$\sqrt{y}=\dfrac{-x}{x-1}$, $\boxed{y=\left(\dfrac{-x}{x-1}\right)^2}$

b) $f(x)=\dfrac{3}{2+\log x}$

$y=\dfrac{3}{2+\log x}$

$x(2+\log y)=3$

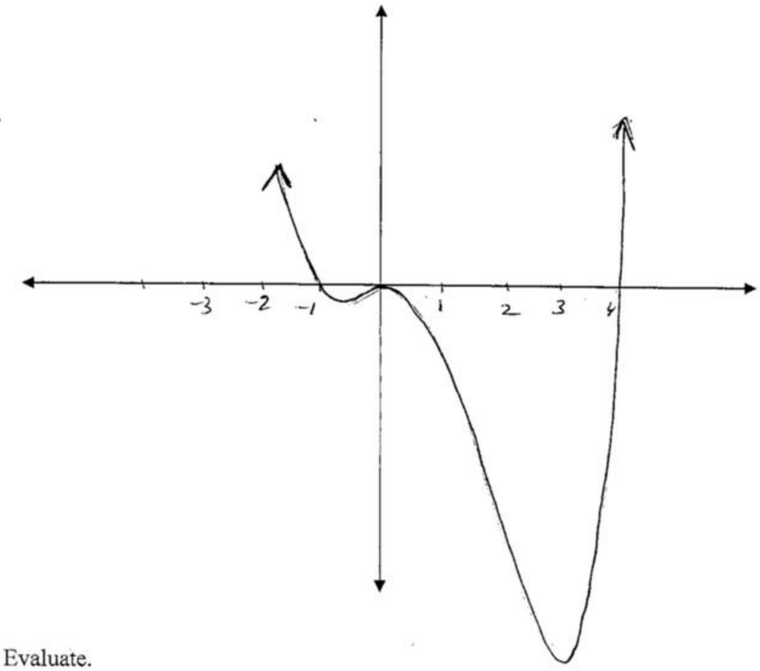
$2x+x\cdot\log y=3$

1) $x=\dfrac{3}{2+\log y}$

$\log y=\dfrac{3-2x}{x}$

$\boxed{y=10^{\frac{3-2x}{x}}=f^{-1}(x)}$

9. Sketch the graph of $y=\left(x^2+1\right)x^3\left(x^3-3x^2-4x\right)$. Label any x or y intercepts.
[10 points]



10. Evaluate.
[9 points]

a) $\tan^{-1}\left(\sqrt{3}\right)$, Since $\tan(60°)=\sqrt{3}$, then $\tan^{-1}(\sqrt{3})=60°$.

b) $\sin\left(\frac{11\pi}{6}\right) = \sin(330°) = -\sin(30°) = -\frac{1}{2}$.

c) $\cos^2\left(\frac{7\pi}{6}\right)$, $\cos(210°) = -\cos(30°) = -\frac{\sqrt{3}}{2}$

$\cos^2\left(\frac{7\pi}{6}\right) = \left(-\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$.

11. Find the half-life of Tritium, which decays at a rate of 5.4% per year.
[6 points]

$Q(t) = Q_0 e^{-0.054t}$

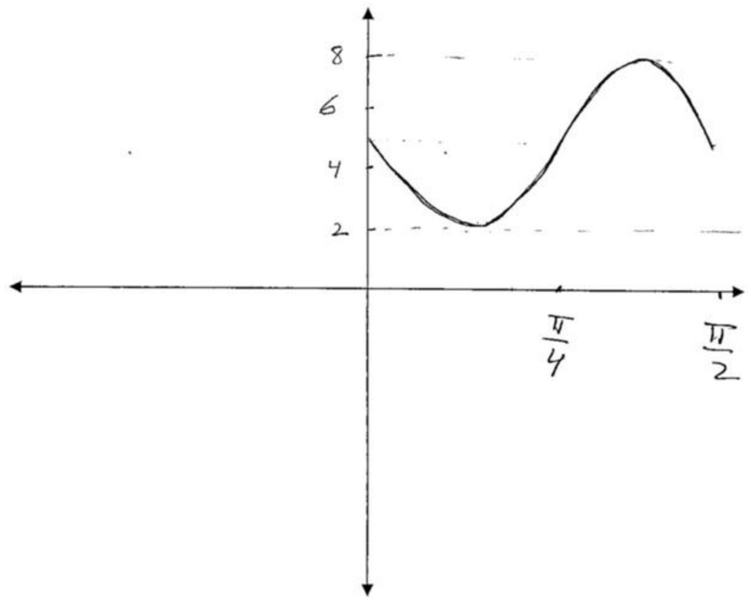
Find t such that $Q(t) = \frac{1}{2}Q_0$

$Q_0 e^{-0.054t} = \frac{1}{2}Q_0$

$(-0.054)t = \ln(0.5)$

$t = \ln(0.5)/-0.054$.

12. Sketch one cycle of the graph $y=-3\sin(4\theta)+5$. Label 5 points on the graph.
[10 points]



13. Given the sum and difference formulas $\sin(A-B)=\sin A\cos B-\cos A\sin B$ and $\cos(A+B)=\cos A\cos B-\sin A\sin B$. Find:
[8 points]

a) $\sin\left(-\frac{\pi}{12}\right) = \sin(-15^\circ) = -\sin(15^\circ) =$

$-\sin(45^\circ-30^\circ) = -\left[\sin(45)\cdot\cos(30) - \cos(45^\circ)\cdot\sin(30)\right]$

b) $\cos\left(\frac{5\pi}{12}\right) = \cos(75^\circ) = \cos(45^\circ+30^\circ) =$

$\cos(45^\circ)\cdot\cos(30^\circ) - \sin(45^\circ)\cdot\sin(30^\circ) =$

$\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

→ 14. Find all real numbers that satisfy each equation.
[28 points]

$\boxed{\text{in } [0, 2\pi] \text{ only}}$

Given: $\sin 2x = 2\cos x\sin x$

$\cos 2x = \cos^2 x - \sin^2 x = \underbrace{2\cos^2 x - 1}_{\uparrow} = 1 - 2\sin^2 x$

a) $\cos 2x + \cos x = 0$

$2\cos^2(x) - 1 + \cos x = 0 = 2\cos^2(x) + \cos(x) - 1.$

$(2\cos(x) - 1)(\cos(x) + 1) = 0$

i) $2\cos(x) = 1$, $\cos(x) = \frac{1}{2}$ | $x = 60^\circ$, $x = -60^\circ$ | ii) $\cos x = -1$, $x = 180^\circ$

b) $2\cos^2\theta = 3\sin\theta + 3$, $2(1-\sin^2\theta) = 3\sin\theta + 3$

$2\sin^2\theta + 3\sin\theta + 1 = 0$, $(2\sin\theta + 1)(\sin\theta + 1) = 0$

• $\sin\theta = -\frac{1}{2}$, • $\sin\theta = -1$ $\boxed{\theta = 270^\circ}$ $\boxed{\theta = 210^\circ \quad \theta = 330^\circ}$

c) $\sin 2x = \sin x$, $2\sin x \cdot \cos x - \sin x = 0$

$\sin(x)[2\cos(x) - 1] = 0$

i) $\sin(x) = 0$, $\boxed{x = 0, \pi, 2\pi}$

ii) $\cos(x) = \frac{1}{2}$ $\boxed{x = 60^\circ, -60^\circ}$

d) $5(2x+3)^3(x+2)^4(9x-3)^5 = 0$

$x = \frac{-3}{2}, -2, \frac{1}{3}$

e) $12 \cdot 7^x = 4 \cdot 9^{x+2}$, $7^x = \frac{4}{12} \cdot 9^{x+2} = \frac{1}{3} \cdot 9^{x+2}$

$x \cdot \ln(7) = \ln(\frac{1}{3}) + (x+2)\ln(9)$

$x[\ln(7) - \ln(9)] = 2\ln(9) - \ln(3)$

$x = \frac{2\ln(9) - \ln(3)}{\ln(7) - \ln(9)}$

f) $\log(5x+10) = 3$

$10^3 = 5x + 10$, $5x = 10^3 - 10$

$x = \frac{10^3 - 10}{5} = 198.$

g) $7e^{3x} + 2e^{3x} = 2$

$e^{3x}[7+2] = 2$, $e^{3x} = \frac{2}{9}$, $3x = \ln(2/9)$

$\boxed{x = \frac{\ln(2/9)}{3}}$

15. Prove that the following equation is an identity.
[5 points]

$1 + \sec x \sin x \tan x = \sec^2 x$

$1 + \frac{1}{\cos(x)} \cdot \sin(x) \cdot \frac{\sin(x)}{\cos(x)} = 1 + \frac{\sin^2 x}{\cos^2 x}$

$= 1 + \tan^2(x) = \sec^2 x$

16. Simplify each expression.
[20 points]

a) $\sin^2(5\theta) + \cos^2(5\theta) = \underline{1}$

b) $(1-\cos\theta)(1+\cos\theta)$ $= 1+\cos\theta - \cos\theta - \cos^2\theta$

$= 1-\cos^2\theta = \sin^2\theta$

c) $-\cos(-\theta)\cdot\tan(-\theta)$ $= -\cos(\theta)\cdot\dfrac{\sin(-\theta)}{\cos(-\theta)} = +\dfrac{\cos\theta\cdot\sin\theta}{\cos\theta}$

$= \sin\theta$

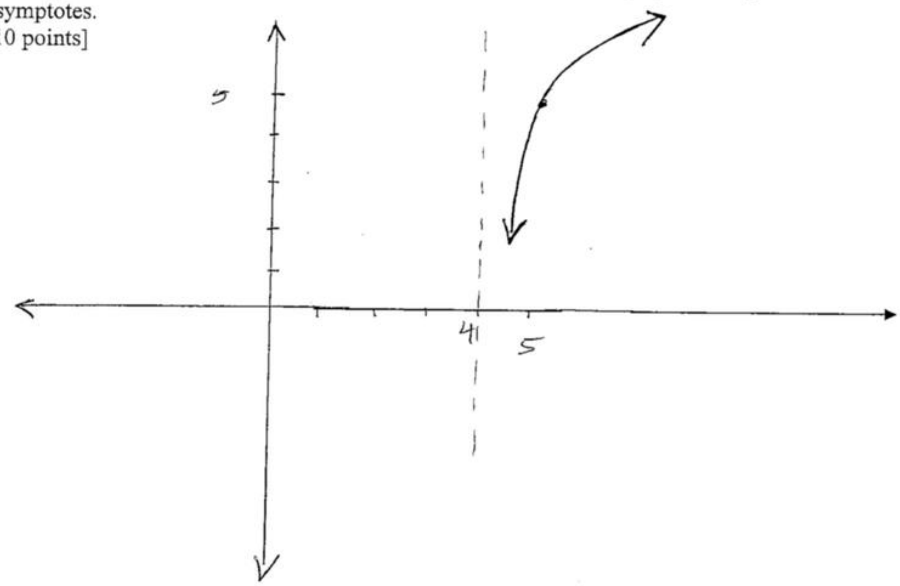
d) $\dfrac{\sin^2 x-\sin x-2}{\sin^2 x-4}$ $= \dfrac{(\sin x+1)(\sin x-2)}{(\sin x-2)(\sin x+2)}$

$= \dfrac{\sin x+1}{\sin x+2}$

e) $\dfrac{\sin x}{\csc x}+\dfrac{\cos x}{\sec x}$ $= \dfrac{\sin x}{\frac{1}{\sin x}} + \dfrac{\cos x}{\frac{1}{\cos x}} = \sin^2 x+\cos^2 x$

$= 1.$

17. Sketch the graph of $y=3\log_2(x-4)+5$. Label 3 points on the graph and any asymptotes.
[10 points]



18. What is the value (if any) of the following?
[14 points]

a) $\ln(x)$ *as* $x\to 0^-$, $\lim_{x\to 0^-}\ln x$ [$x>0$. We cannot approach zero from the left.

b) $\log_5(x-2)$ *as* $x\to 2^+$, $\lim_{x\to 2^+}\log_5(x-2) = -\infty$

c) x^{-3} as $x \to 0^-$, $\lim_{x \to 0^-} \frac{1}{x^3} = -\infty$

d) $\frac{25}{x^2}$ as $x \to -\infty$, $\lim_{x \to -\infty} \frac{25}{x^2} = 0$

e) e^{3x} as $x \to -\infty$, $\lim_{x \to -\infty} e^{3x} = 0$

f) $\sqrt[4]{x}$ as $x \to 0^-$, $\lim_{x \to 0^-} \sqrt[4]{x}$ Cannot approach zero from the left. "Does not exist"

g) $-10x^5$ as $x \to -\infty$

$$\lim_{x \to -\infty} (-10x^5) = \infty$$

19. Find the cubic function with x intercepts $(3,0)$, $\left(-\frac{1}{2}, 0\right)$, $(4,0)$ and y intercept $(0,3)$.
[5 points]

$y = k(x-3)(x+\frac{1}{2})(x-4)$. To find $\underline{k}$:

$3 = k(-3)(\frac{1}{2})(-4) = 6k \Rightarrow \boxed{k = \frac{1}{2}}$

20. How many zeros does each of the following relations have?.
[10 points)

a) $x = 3$, one

b) $y = (x-1)^2(x-5)(x-7)^9$, $x = 1$, multiplicity 2
$x = 5$, " 1
$x = 7$, " 9

c) $y = 4x^2 + 1$

no real zeros

d) $y = 4x^2 - 1 = (2x+1)(2x-1)$
Two zeros

e) $(x+2)(4x^2+1)(x^2+x+1)$

$x = -2$, multiplicity 1

21. A population of animals oscillates between a high of 2000 on January 1 ($t = 0$) and a low of 500 on July 1 ($t = 6$).
[8 points]

a) Find a formula for the population, P, in terms of the time, t, in months.

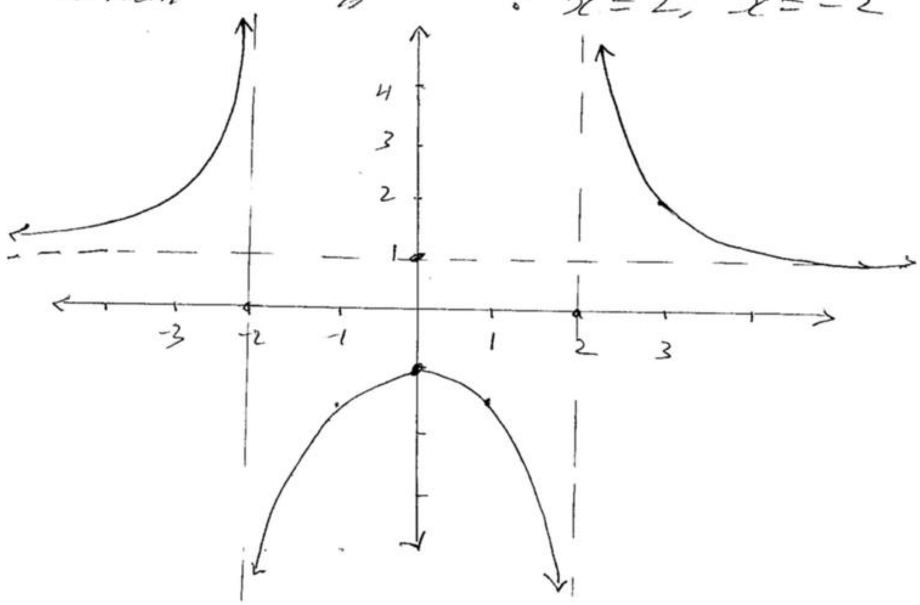
$P(t) = (750)\cos(\frac{\pi}{6}t) + 1250$.

b) Interpret the amplitude and period of the function $P = f(t)$.

Amplitude : 750
period : $\frac{2\pi}{12} = \pi/6$.

Bonus: Sketch the graph of $y = \frac{x^2+4}{x^2-4}$. Clearly label any asymptotes and intercepts.
[5 points]

horizontal asymptotes: $y = 1$
vertical " : $x = 2,\ x = -2$



Range : $(-\infty, -1] \cup (1, \infty)$.